第10回 代数群と量子群の表現論
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EXCITED YOUNG DIAGRAMS AND EQUIVARIANT SCHUBERT CALCULUS FOR GRASSMANNIANS

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Summary

In this report we explain briefly the results of the paper [I-N] about the torus-equivariant cohomology of Grassmannians of type B,C,D.

1. Notations

We slightly change the notation from [I-N] so that the inclusion $W(X_n) \subset W(X_{n+1})$ of Weyl group becomes natural for $X = B, C, D$. For the simple roots we adopt the following conventions.

type B_n: $\alpha_0 = t_1, \alpha_i = t_{i+1} - t_i, (i = 1, ..., n-1)$.

type C_n: $\alpha_0 = 2t_1, \alpha_i = t_{i+1} - t_i, (i = 1, ..., n-1)$.

type D_n: $\alpha_{\hat{1}} = t_2 + t_1, \alpha_i = t_{i+1} - t_i, (i = 1, ..., n-1)$.

The Weyl groups are subgroup of permutations of $2n$ letters $\bar{n} < \cdots < \bar{1} < 1 < \cdots < n$, such that $W(B_n) = W(C_n) =< s_0, s_1, ..., s_{n-1} >$ and $W(D_n) =< s_{\hat{1}}, s_1, ..., s_{n-1} >$, where $s_0 = (\bar{1}, 1), s_i = (i, i+1)(\bar{i}, \overline{i+1})$ $(1 \le i \le n-1)$, and $s_{\hat{1}} = (1, \bar{2})(2, \bar{1})$. We use one line notation $w = [w(1), w(2), ..., w(n)]$. For type D_n, the barred numbers appear even times. Grassmannian permutations are $w = [w_1, w_2, ..., w_n]$ such that $w_1 < w_2 < \cdots < w_n$. If the barred part of a Grassmannian permutation v is $\overline{b_1}, ..., \overline{b_r}$, we set $\lambda_v = (b_1, ..., b_r)$ for type B_n and C_n, while $\lambda_v = (b_1 - 1, ..., b_r - 1)$ for type D_n. cf.[B-H].

2. Introduction

Let G be a semisimple algebraic group with parabolic subgroup P which contains a maximal torus T. T acts on the homogeneous space G/P by left translation. Let W be the Weyl group of G, and W_P be the subgroup corresponding to P. The T-fixed points are parametrized by the coset $W^P := W/W_P$ of Weyl group W of G by the subgroup W_P corresponding to P. For $v \in W/W_P$, $e(v) := vP/P$ is T-fixed point and $(G/P)^T = \{e(v)\}_{v \in W^P}$. Let $X_w := \overline{B_- e(w)} \subset G/P$ be the Schubert variety corresponding to $w \in W^P$ and opposite Borel subgroup B_-.

We want to describe equivariant Schubert class $[X_w] \in H_T^*(G/P)$ for the cases

- type B_n : $SO(2n+1)/P_0$
- type C_n : $Sp(2n)/P_0$
- type D_n : $SO(2n)/P_{\hat{1}}$.

Here P_i is the maximal parabolic subgroup corresponding to the root α_i.

We do not explain the equivariant cohomology and equivariant Schubert class in detail here. A good introduction is a paper of Tymoczko [Ty]. We use the localization map to describe equivariant cohomology. The inclusion map of the fix points $\iota : (G/P)^T \hookrightarrow G/P$ induces the pullback homomorphism

$$\iota^* : H_T^*(G/P) \longrightarrow H_T^*((G/P)^T) = \bigoplus_{v \in W^P} H_T^*(e_v)$$

which is known to be injective. This map ι^* as well as each of the component map $\iota_v^* : H_T^*(G/P) \to H_T^*(e_v)$ is called the localization map. As a result, by the work of Arabia [Ar], the image $\iota^*([X_w])$ can be identified with Kostant and Kumar's ξ-function [K-K], i.e.

$$[X_w]|_v = \iota_v^*([X_w]) = \xi^w(v).$$

There is a formula of Billey (also of Andersen et al.) for $\xi^w(v)$ using reduced expression of v.

Proposition 1. *([A-J-S], [Bi]) Let $w, v \in W$ such that $w \leq v$. Fix a reduced expression $s_{i_1} \cdots s_{i_k}$ for v. Put $\beta_t = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t})$ for $1 \leq t \leq k$. Then*

$$\xi^w(v) = \sum_{j_1,\ldots,j_s} \beta_{j_1} \cdots \beta_{j_s},$$

where the sum is over all sequences $1 \leq j_1 < \cdots < j_s \leq k$ such that $s_{i_{j_1}} \cdots s_{i_{j_s}}$ is a reduced expression for w.

3. Main results

We display here main results of the paper [I-N] which are extensions of the results of Ikeda [Ik] for the case of type C_n. The relation to known results are explained in detail in that paper and we omit the references to known results here.

There are two kinds of formulas to express the localization $[X_w]|_v$. For combinatorial formulas, we have

Theorem 1. *([I-N;Theorem 3])*

$$B_n : [X_w]|_v = \sum_{C \in \mathcal{E}^B_{\lambda_v}(\lambda_w)} \prod_{(i,j)\in C} 2^{-\delta_{i,j}}(t_{v(\bar{i})} - t_{v(j)})$$

$$C_n : [X_w]|_v = \sum_{C \in \mathcal{E}^C_{\lambda_v}(\lambda_w)} \prod_{(i,j)\in C} (t_{v(\bar{i})} - t_{v(j)})$$

$$D_n : [X_w]|_v = \sum_{C \in \mathcal{E}^D_{\lambda_v}(\lambda_w)} \prod_{(i,j)\in C} (t_{v(\bar{i})} - t_{v(j+1)})$$

Here $\mathcal{E}^X_\lambda(\mu)$ is the set of *excited Young diagrams* of type $X = B, C, D$ explained in section 4. We also use the convention that $t_{\bar{i}} = -t_i$ for $i > 0$.

The closed formulas using factorial Schur P, Q- functions which are explained in section 5 are as follows. For Grassmannian permutation $v = [\overline{b_1}, \ldots, \overline{b_r}, v_{r+1}, \ldots, v_n]$ with $v_{r+1} > 0$, we denote $t_v = (t_{b_1}, \ldots, t_{b_r}, 0, \ldots, 0)$. Then we have

Theorem 2. *([I-N;Theorem 4])*

$$B_n : [X_w]|_v = P_{\lambda_w}(t_v|0, t).$$

$$C_n : [X_w]|_v = Q_{\lambda_w}(t_v|0, t).$$

$$D_n : [X_w]|_v = P_{\lambda_w}(t_v|t).$$

Using the Pfaffian formula for factorial Schur P, Q-functions, or Gessel-Viennot type combinatorial arguments for excited Young diagrams, we have

Corollary 1. *(Equivariant Giambelli) ([I-N; Cor.2])*
In the cohomology ring $H_T^*(G/P)$ of type B, C, D Grassmannians, we have

$$[X_\lambda] = \mathrm{Pf}\left([X_{\lambda_i,\lambda_j}]\right)_{1 \leq i < j \leq r_0(\lambda)},$$

where we denote by X_λ the Schubert variety corresponding to λ.

Here $r_0(\lambda)$ is the number of parts in λ with convention that if the number of parts λ is odd, we add part 0 so that $r_0(\lambda)$ is always even.

For multiplicity of singular points, we have

Proposition 2. *([I-N; Prop.12]) Let G/P be either of types C_n, D_n and $w \leq v \in W^P$. The value of $[X_w]|_v$ evaluated at a vector $h_v \in Lie(T)$ (which makes each summand of Theorem 1 to be 1) gives the multiplicity $m_v(X_w)$ of the variety X_w at e_v.)*

Corollary 2. *([I-N; Cor.4]) Let G/P be the Grassmannian of types C_n or D_n. Let $w \le v \in W^P$, and $\lambda \le \mu$ be the corresponding strict partitions. Then we have*

(1) $$m_v(X_w) = \sharp\mathcal{E}_\mu^C(\lambda) \text{ for type } C_n, \qquad m_v(X_w) = \sharp\mathcal{E}_\mu^D(\lambda) \text{ for type } D_n.$$

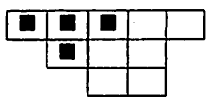
(2) $$m_v(X_w) = \mathrm{Pf}(m_v(X_{\lambda_i,\lambda_j}))_{1\le i<j\le r_0(\lambda)}.$$

Remark Recently Raghavan and Upadhyay [R-U] obtained essentially the same formula as (1) for type D_n in quite different approach.

4. Excited Young diagrams

The idea of excited Young diagrams was inspired by the work of Lakshmibai et al. [L-R-S] in which the localization of equivariant cohomology is described using nonintersecting paths on a Young diagram for type A Grassmannians. To relate the determinant (for type A) and Pfaffian (for type B, C, D) formula directly, we use the "complement" of the nonintersecting paths of [L-R-S]. Our excited Young diagram is also described as nonintersecting paths and fits the formula of Billey [Bi](Prop. 1) and fully commutativity of Grassmannian permutations cf. [St]. Similar object is also obtained by Kreiman [Kr] independently to us.

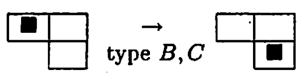
We consider the excited Young diagrams for the cases of type B, C, D Grassmannians. There are other versions or generalizations but we do not explain here. Let $\mathcal{SP}(n)$ be the set of strict partitions of length $\le n$ i.e. $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0) \in \mathcal{SP}(n)$ such that $r \le n$. We consider a pair of strict partitions $\lambda \subset \mu \in \mathcal{SP}(n)$ and their shifted Young diagrams. For example, $\lambda = (3,1)$ and $\mu = (5,3,2)$ the shifted Young diagrams are as below.

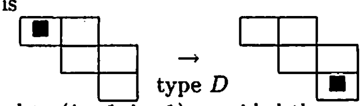


We put λ (black boxes) inside μ with overriding the left top cell in the position $(1,1)$ for usual matrix index. This is the ground state of λ, and we move the cells of λ (also called boxes) inside μ according to the rule defined as follows.

We first define elementary excitations. There are two types of cells in μ, diagonal and off-diagonal. Diagonal cell is a cell in the position (i,i), and off-diagonal cell is in the position (i,j) for $i<j$.

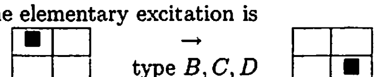
For a box in diagonal cell, the elementary excitation is





i.e. for type B, C, a box in the position (i,i) is moved to $(i+1, i+1)$ provided the positions $(i, i+1), (i+1, i+1)$ are vacant, and for type D, the box in the position (i,i) is moved to $(i+2, i+2)$ provided the positions $(i, i+1), (i+1, i+1), (i+1, i+2), (i+2, i+2)$ are vacant. All the rest of the cells (black boxes) do not change their positions.

For a box in off-diagonal cell, the elementary excitation is



i.e. a box in the position (i,j) is moved to the position $(i+1, j+1)$, provided the positions $(i+1, j), (i, j+1), (i+1, j+1)$ are vacant, and all the rest boxes are unchanged.

For subsets $C, C' \subset \mu$, we denote $C \xrightarrow{X} C'$ if C' is obtained from C by an elementary excitation of type X by a box in C. Then we define the set of excited Young diagrams $\mathcal{E}_\mu^X(\lambda)$ as the set of diagrams $C \subset \mu$, which are obtained by successive applications of elementary excitation of type X, i.e. for each $X = B, C, D$,

$$\mathcal{E}_\mu^X(\lambda) := \{C \subset \mu \mid \lambda = C_0 \xrightarrow{X} C_1 \xrightarrow{X} \cdots \xrightarrow{X} C_k = C\}.$$

Examples for $\mathcal{E}^C_{(5,3,2)}(3,1)$ and $\mathcal{E}^D_{(5,4,2,1)}(3,1)$ are in section 6.

5. Factorial Schur Q-functions

Let $a_1, a_2, \ldots$ be indeterminates. The factorial power $(x|a)^k$ is defined by $(x-a_1)(x-a_2)\cdots(x-a_k)$.

Definition 1. *[Iv] Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{SP}(n)$. Put*

$$(5.1) \qquad P_\lambda(x|a) = \frac{1}{(n-r)!}\sum_{w \in S_n} w\left((x_1|a)^{\lambda_1}\cdots(x_r|a)^{\lambda_r}\prod_{1\le i\le r,\ i<j\le n}\frac{x_i+x_j}{x_i-x_j}\right),$$

where $w \in S_n$ acts as a permutation of variables $x_1, \ldots, x_n$. We also put $Q_\lambda(x|a) = 2^r P_\lambda(x|a)$.

Remark

In [Iv] Ivanov assumed $a_1 = 0$. In type D_n case , we must use a_1 term. $P_\lambda(x|a)$ has mod 2 stability [I-N Prop.8], but barred numbers of $v \in W(D_n)$ always appear even times, so we assume that the number of variables x_i is always even.

Factorial Schur P (and Q) - functions have important vanishing property which may be a good reason for describing equivariant Schubert classes. For strict partition $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$ with $r \le n$, define an n-tuple

$$a_\lambda = \begin{cases}(a_{\lambda_1+1}, \ldots, a_{\lambda_r+1}, 0, \ldots, 0) & \text{if } r \text{ is even}\\ (a_{\lambda_1+1}, \ldots, a_{\lambda_r+1}, a_1, 0, \ldots, 0) & \text{if } r \text{ is odd}\end{cases}.$$

Proposition 3. *[Iv] We have $P_\lambda(a_\mu|a) = 0$ unless $\mu \ge \lambda$.*

The following Pfaffian formula is also important for describing equivariant Schubert classes.

Proposition 4. *[Iv] For a strict partition $\lambda \in \mathcal{SP}$ we have*

$$P_\lambda(x|a) = \mathrm{Pf}(P_{\lambda_i,\lambda_j}(x|a))_{1\le i<j\le r_0(\lambda)}.$$

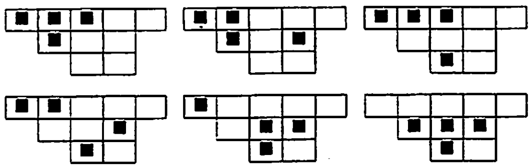
6. Examples

Examples for Theorem 1

C_5

$w = [\bar{3}, \bar{1}, 2, 4, 5], \quad v = [\bar{5}, \bar{3}, \bar{2}, 1, 4].$

	$\bar{5}$	$\bar{3}$	$\bar{2}$	1	4
5	$2t_5$	t_5+t_3	t_5+t_2	t_5-t_1	t_5-t_4
3		$2t_3$	t_3+t_2	t_3-t_1	
2			$2t_2$	t_2-t_1	

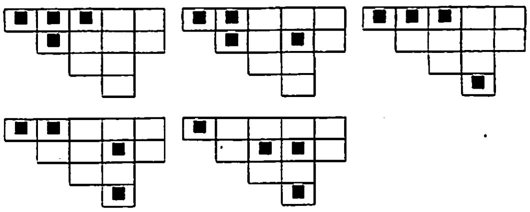


$$\begin{aligned}[X_w]|_v = {}& 2t_5 2t_3(t_5+t_3)(t_5+t_2) + 2t_5 2t_3(t_5+t_3)(t_3-t_1) + 2t_5 2t_2(t_5+t_3)(t_5+t_2)\\ &+ 2t_5 2t_2(t_5+t_3)(t_3-t_1) + 2t_5 2t_2(t_3+t_2)(t_3-t_1) + 2t_3 2t_2(t_3+t_2)(t_3-t_1).\end{aligned}$$

D_6

$w = [\bar{4}, \bar{2}, 1, 3, 5, 6], v = [\bar{6}, \bar{5}, \bar{3}, \bar{2}, 1, 4].$

	$\bar{5}$	$\bar{3}$	$\bar{2}$	1	4
6	t_6+t_5	t_6+t_3	t_6+t_2	t_6-t_1	t_6-t_4
5		t_5+t_3	t_5+t_2	t_5-t_1	t_5-t_4
3			t_3+t_2	t_3-t_1	
2				t_2-t_1	



$$[X_w]|_v = (t_6+t_5)(t_6+t_3)(t_6+t_2)(t_5+t_3)+(t_6+t_5)(t_6+t_3)(t_5+t_3)(t_5-t_1)$$
$$+(t_6+t_5)(t_6+t_3)(t_6+t_2)(t_2-t_1)+(t_6+t_5)(t_6+t_3)(t_5-t_1)(t_2-t_1)+(t_6+t_5)(t_5+t_2)(t_5-t_1)(t_2-t_1).$$

Examples for Theorem 2
type C_3
$w=[\bar{2},1,3], v=[\bar{3},\bar{2},1]$.
In this case $\lambda_w=(2)$ and $t_v=(t_3,t_2,0)$.
$Q_2(x_1,x_2,x_3|a)=2(x_1+x_2+x_3)(x_1+x_2+x_3-a_2)$ and
$[X_w]|_v=Q_2(t_3,t_2|0,t_1)=2(t_3+t_2)(t_3+t_2-t_1)$.

type D_5
$w=[\bar{4},\bar{2},1,3,5], v=[\bar{5},\bar{3},1,2,4]$.
In this case $\lambda_w=(3,1)$ and $t_v=(t_5,t_3,0,0)$.
$P_{3,1}(x_1,x_2|a)=(x_2+x_1)(x_2-a_1)(x_1-a_1)(x_1+x_2-a_2-a_3)$
and
$[X_w]|_v=P_{3,1}(t_5,t_3|t_1,t_2)=(t_5+t_3)(t_5-t_1)(t_3-t_1)(t_5-t_2)$.

7. Further comments

After the work [I-N], T.Ikeda, L. Mihalcea and the author get progress to these subjects. Above all, we find a geometric meaning of factorial Schur P,Q-functions and extended the Grassmannian cases to the full flag varieties G/B, by defining new double Schubert polynomials. The details will be explained elsewhere.

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