Characters of Party algebras

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Complex reflection group
$W_{r,k} := G(r,1,k)$ acts on $V = \mathbf{C}^k$

Party algebra $P_{n,r}(k)$:=$\mathrm{End}_{W_{r,k}}(V^{\otimes n})$

Problem

- Define a notion of "Conjugacy class" and character table.
- Find an inductive algorithm to calculate character values. (Murnaghan-Nakayama rule)

(for $r = 1$ T.Halverson(2001)J. of Algebra)

Schur-Weyl duality

$$V^{\otimes n} = \bigoplus_{\lambda} V^{\lambda} \otimes S^{\lambda}$$

where
V^{λ} is an irreducible $W_{r,k}$-module, and
S^{λ} is an irreducible $P_{n,r}(k)$-module.

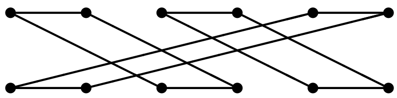
The index $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(r-1)})$ is an r-tuple of partitions which is in $\widehat{W}_{r,k}$ i.e. $|\lambda| := \sum_{i=0}^{r-1} |\lambda^{(i)}| = k$, and satisies $||\lambda^*|| \leq n$ and $||\lambda^*|| \equiv n \pmod r$, where $||\lambda^*|| := \sum_{i=1}^{r-1} i|\lambda^{(i)}| + r|(\lambda^{(0)})^*|$ and $(\lambda^{(0)})^*$ is the partition obtained by deleting the first row of $\lambda^{(0)}$.

Let $\Lambda_{n,r} := \left\{ \mu = (\mu^{(1)}, \cdots, \mu^{(r)});\ \begin{array}{l} \mu^{(i)} \text{ is a partiton,} \\ ||\mu|| \le n, \text{ and} \\ ||\mu|| \equiv n \pmod{r} \end{array} \right\}$

where $||\mu|| := \sum_{i=1}^{r} i|\mu^{(i)}|$.

Standard cyclic permutation

$\gamma_\ell^{[t]} \in \mathrm{End}(V^{\otimes t\ell})$ of thickness t and length ℓ.

e.g. $\gamma_3^{[2]} =$  $\in \mathrm{End}(V^{\otimes 6})$

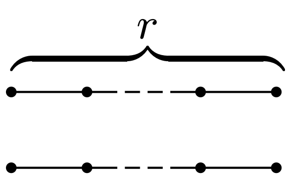
For a partition $a = (a_1, a_2, \cdots, a_s)$, define

$\gamma_a^{[t]} := \gamma_{a_1}^{[t]} \otimes \gamma_{a_2}^{[t]} \otimes \cdots \otimes \gamma_{a_s}^{[t]} \in \mathrm{End}(V^{\otimes t|a|})$.

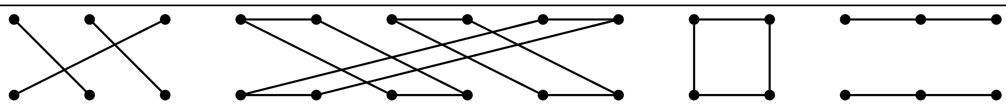
Standard elements of $P_{n,r}(k)$

For $\mu \in \Lambda_{n,r}$, define $\boxed{d_{\mu,n} := \gamma_\mu \otimes e_r^{\otimes \frac{n-||\mu||}{r}}}$

where $\gamma_\mu = \gamma^{[1]}_{\mu^{(1)}} \otimes \gamma^{[2]}_{\mu^{(2)}} \otimes \cdots \otimes \gamma^{[r]}_{\mu^{(r)}} \in \mathrm{End}(V^{\otimes ||\mu||})$

and $e_r =$  $\in \mathrm{End}(V^{\otimes r})$.

e.g. $n = 14, r = 3, \mu = (3, 31, 0)$ then $||\mu|| = 11$,

$d_{(3,31,0),14} =$ 

Define a **class function** $f_\ell^{[t]}$ on $W_{r,k}$ by

$$f_\ell^{[t]}(w_\rho) := \sum_{i=0}^{r-1} \sum_{d|\ell} d\, m_d(\rho^{(i)}) \zeta^{it\frac{\ell}{d}}$$

where $w_\rho \in W_{r,k}$ has cycle type $\rho = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(r-1)})$, $m_d(\rho^{(i)})$ is the multiplicity of parts of size d in $\rho^{(i)}$, and $\zeta = e^{\frac{2\pi\sqrt{-1}}{r}}$ is the primitive r-th root of 1.

Lemma 1 For $w \in W_{r,k}$, the bitrace is

$$btr(w, \gamma_\ell^{[t]}) = f_\ell^{[t]}(w) \quad \text{on } V^{\otimes t\ell}$$

$$\text{and} \quad btr(w, e_r) = k \quad \text{on } V^{\otimes r}.$$

For a partition $a = (a_1, a_2, \cdots, a_s)$, define

$$f_a^{[t]}(w) := f_{a_1}^{[t]}(w) f_{a_2}^{[t]}(w) \cdots f_{a_s}^{[t]}(w).$$

Theorem 1

For $w \in W_{r,k}$, $\mu \in \Lambda_{n,r}$

$$btr(w, d_{\mu,n}) = k^{\frac{n-||\mu||}{r}} f_\mu(w)$$

where $f_\mu(w) := f_{\mu^{(1)}}^{[1]}(w) \times f_{\mu^{(2)}}^{[2]}(w) \times \cdots \times f_{\mu^{(r)}}^{[r]}(w)$

Corollary 1(Frobenius formula)

$$k^{\frac{n-||\mu||}{r}} f_\mu(w) = \sum_\lambda \chi^\lambda_{W_{r,k}}(w)\chi^\lambda_{P_{n,r}(k)}(d_{\mu,n})$$

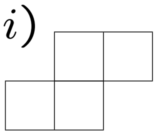
Corollary 2

$$\chi^\lambda_{P_{n,r}(k)}(d_{\mu,n}) = < k^{\frac{n-||\mu||}{r}} f_\mu \, , \, \chi^\lambda_{W_{r,k}} >$$

Murnaghan-Nakayama rule for $W_{r,k}$

$$\chi^\lambda_{W_{r,k}}(w_\rho) = \sum_{i=0}^{r-1} \sum_{\lambda^{-\ell} \subset \lambda} (-1)^{ht(\lambda/\lambda^{-\ell})} \zeta^{ij} \chi^{\lambda^{-\ell}}_{W_{r,k-\ell}}(w_{\bar\rho})$$

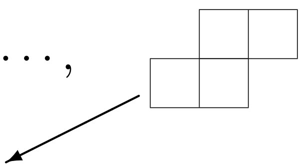
where $\bar\rho$ is obtained from ρ by deleting a cycle of length ℓ from $\rho^{(j)}$, and $\lambda/\lambda^{-\ell}$ is a border strip of length ℓ in position (i).

e.g. $\lambda/\lambda^{-\ell} = (\emptyset, \emptyset, \cdots, \emptyset,$ (i)  $, \emptyset, \cdots, \emptyset)$, $ht(\lambda/\lambda^{-\ell}) = 1$

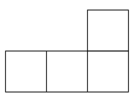
Proposition For $\delta, \lambda \in \widehat{W}_{r,k}$

$$< f^{[t]}_\ell \chi^\delta_{W_{r,k}}, \chi^\lambda_{W_{r,k}} > = \sum_{d|\ell} \sum_{\lambda^{-d} \subset \lambda, \lambda^{-d} \subset \delta} (-1)^{ht(\delta/\lambda^{-d})} (-1)^{ht(\lambda/\lambda^{-d})}$$

where λ^{-d} runs through $\widehat{W}_{r,k-d}$ such that e.g.

$\lambda/\lambda^{-d} = (\emptyset, \cdots, \quad \emptyset \; , \cdots,$  $, \cdots, \emptyset)$

shift $t\frac{\ell}{d}$ (mod r)

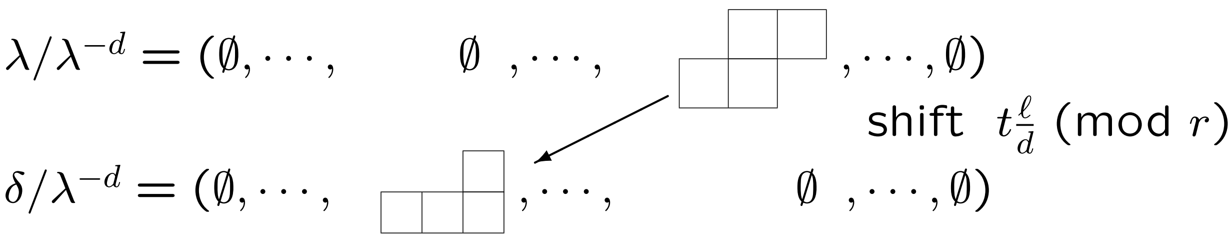
$\delta/\lambda^{-d} = (\emptyset, \cdots,$  $, \cdots, \quad \emptyset \; , \cdots, \emptyset)$

Murnaghan-Nakayama rule for $P_{n,r}(k)$

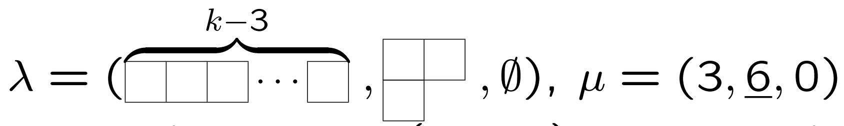
$$\chi^{\lambda}_{P_{n,r}(k)}(d_{\mu,n}) = \sum_{d|\ell} \sum_{\lambda^{-d}\subset\lambda} \left(\sum_{\lambda^{-d}\subset\delta} (-1)^{ht(\lambda/\lambda^{-d})} (-1)^{ht(\delta/\lambda^{-d})} \chi^{\delta}_{P_{\bar{n},r}(k)}(d_{\bar{\mu},\bar{n}}) \right)$$

where $\bar{n} = n - t\ell$ and $\bar{\mu} \in \Lambda_{\bar{n},r}$ is obtained from $\mu \in \Lambda_{n,r}$ by deleting a cycle of thickness t and length ℓ.

δ runs through $\widehat{P_{\bar{n},r}}(k)$ such that e.g.

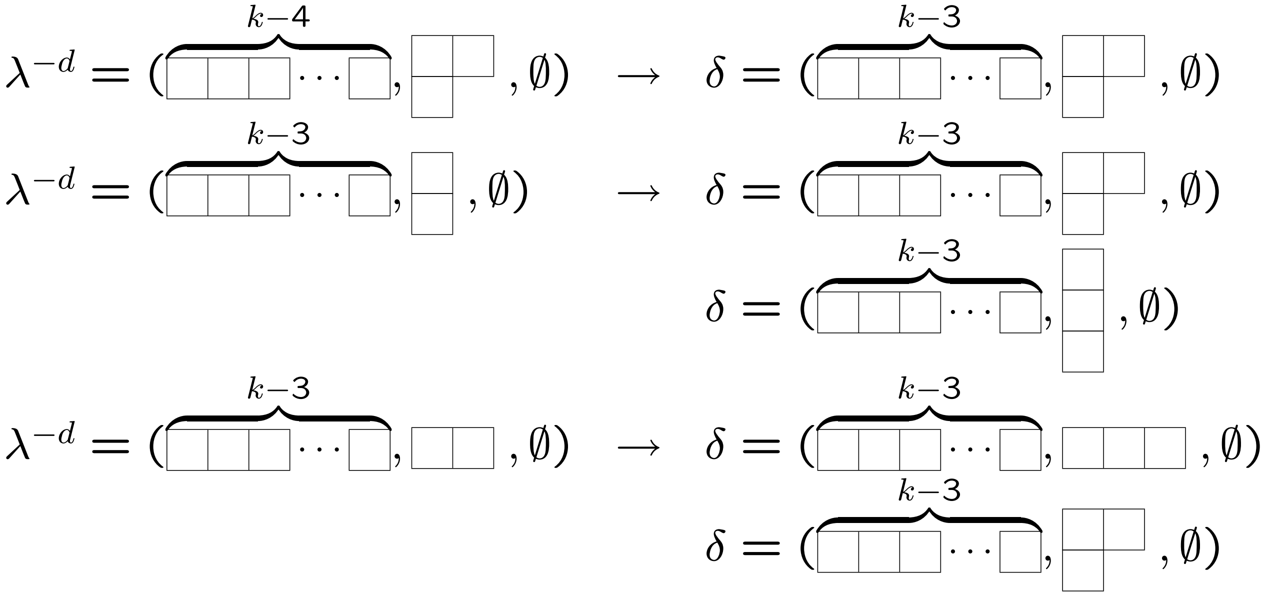


Example $\quad n = 15, r = 3$



$t = 2, \ell = 6,\ \bar{\mu} = (3, 0, 0),\ \bar{n} = n - t\ell = 3$

1) $d = 1,\ t\frac{\ell}{d} = 12 \equiv 0 \pmod{r}$



$$\lambda = (\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \yng(2,1), \emptyset),\ \mu = (3, \underline{6}, 0)$$
$$t = 2, \ell = 6,\ \bar{\mu} = (3,0,0),\ \bar{n} = n - t\ell = 3$$

2) $d = 2,\ t\frac{\ell}{d} = 6 \equiv 0 \pmod{r}$

$$\lambda^{-d} = (\overbrace{\yng(3)\cdots\yng(1)}^{k-5}, \yng(2,1), \emptyset) \ \rightarrow\ \delta = (\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \yng(2,1), \emptyset)$$

3) $d = 3,\ t\frac{\ell}{d} = 4 \equiv 1 \pmod{r}$

$\lambda^{-d} = (\overbrace{\yng(3)\cdots\yng(1)}^{k-6}, \yng(2,1), \emptyset) \ \rightarrow$ no δ appears.

$$\lambda^{-d} = (\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \emptyset, \emptyset) \ \rightarrow\ \delta = (\overbrace{\yng(3)\cdots\yng(1)}^{k}, \emptyset, \emptyset)$$
$$(-1)^{ht(\lambda/\lambda^{-d})} = -1$$

4) $d = 6,\ t\frac{\ell}{d} = 2 \equiv 2 \pmod{r}$

$\lambda^{-d} = (\overbrace{\yng(3)\cdots\yng(1)}^{k-9}, \yng(2,1), \emptyset) \ \rightarrow$ no δ appears.

Therefore the character value is

$$(\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \yng(2,1), \emptyset)\,|_{(3,\underline{6},0)}$$
$$= 4\times(\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \yng(2,1), \emptyset)\,|_{(3,0,0)} + (\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \yng(1,1,1), \emptyset)\,|_{(3,0,0)}$$
$$+ (\overbrace{\yng(3)\cdots\yng(1)}^{k-3}, \yng(3), \emptyset)\,|_{(3,0,0)} - (\overbrace{\yng(3)\cdots\yng(1)}^{k}, \emptyset, \emptyset)\,|_{(3,0,0)}$$

$$= 4 \times (-1) + 1 + 1 - 1 = -3$$

Character tables for $r = 2$

$n = 2$	0	1	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}$	0	0	2	1^2	$\mu^{(1)}$
$k, 0$	k	1	1	1	
$(k-1)1, 0$		1	1	1	
$k-2, 2$			1	1	
$k-2, 1^2$			-1	1	

$n = 3$	0	1	0	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}$	1	1	3	21	1^3	$\mu^{(1)}$
$k-1, 1$	k	2	1	2	4	
$(k-2)1, 1$		1	0	1	3	
$k-3, 3$			1	1	1	
$k-3, 21$			-1	0	2	
$k-3, 1^3$			1	-1	1	

$n = 4$	0	1	0	0	2	1^2	1	1	0	0	0	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}$	0	0	2	1^2	0	0	2	1^2	4	31	2^2	21^2	1^4	$\mu^{(1)}$
$k, 0$	k^2	k	k	k	2	2	2	2	2	1	4	2	4	
$(k-1)1, 0$		k	k	k	1	3	3	3	1	1	3	3	7	
$k-2, 2$			k	k	0	0	1	3	0	1	2	4	10	
$k-2, 1^2$			$-k$	k	0	0	-1	3	0	1	-2	2	10	
$(k-2)2, 0$					1	1	1	1	1	0	3	1	3	
$(k-2)1^2, 0$					-1	1	1	1	-1	0	-1	1	3	
$(k-3)1, 2$							1	1	0	0	2	2	6	
$(k-3)1, 1^2$							-1	1	0	0	-2	0	6	
$k-4, 4$									1	1	1	1	1	
$k-4, 31$									-1	0	-1	1	3	
$k-4, 2^2$									0	-1	2	0	2	
$k-4, 21^2$									1	0	-1	-1	3	
$k-4, 1^4$									-1	1	1	-1	1	

Character tables for $r = 3$

$n=2$				
	0	0	0	$\mu^{(3)}$
	1	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$	0	2	1^2	$\mu^{(1)}$
$k-1, 0, 1$	1	1	1	
$k-2, 2, 0$		1	1	
$k-2, 1^2, 0$		-1	1	

$n=3$							
	0	1	0	0	0	0	$\mu^{(3)}$
	0	0	1	0	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$	0	0	1	3	21	1^3	$\mu^{(1)}$
$k, 0, 0$	k	1	1	1	1	1	
$(k-1)1, 0, 0$		1	1	1	1	1	
$k-2, 1, 1$			1	0	1	3	
$k-3, 3, 0$				1	1	1	
$k-3, 21, 0$				-1	0	2	
$k-3, 1^3, 0$				1	-1	1	

$n=4$												
	0	1	0	0	0	0	0	0	0	0	0	$\mu^{(3)}$
	0	0	2	1^2	1	1	0	0	0	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$	1	1	0	0	2	1^2	4	31	2^2	21^2	1^4	$\mu^{(1)}$
$k-1, 1, 0$	k	2	1	1	1	3	1	2	1	3	5	
$(k-2)1, 1, 0$		1	0	0	0	2	0	1	0	2	4	
$k-2, 0, 2$			1	1	1	1	1	0	3	1	3	
$k-2, 0, 1^2$			-1	1	1	1	-1	0	-1	1	3	
$k-3, 2, 1$					1	1	0	0	2	2	6	
$k-3, 1^2, 1$					-1	1	0	0	-2	0	6	
$k-4, 4, 0$							1	1	1	1	1	
$k-4, 31, 0$							-1	0	-1	1	3	
$k-4, 2^2, 0$							0	-1	2	0	2	
$k-4, 21^2, 0$							1	0	-1	-1	3	
$k-4, 1^4, 0$							-1	1	1	-1	1	

$n=4, r=2$	0	1	0	0	2	1^2	1	1	0	0	0	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}$	0	0	2	1^2	0	0	2	1^2	4	31	2^2	21^2	1^4	$\mu^{(1)}$
$k, 0$	k^2	k	k	k	2	2	2	2	2	1	4	2	4	
$(k-1)1, 0$		k	k	k	1	3	3	3	1	1	3	3	7	
$k-2, 2$			k	k	0	0	1	3	0	1	2	4	10	
$k-2, 1^2$			$-k$	k	0	0	-1	3	0	1	-2	2	10	
$(k-2)2, 0$					1	1	1	1	1	0	3	1	3	
$(k-2)1^2, 0$					-1	1	1	1	-1	0	-1	1	3	
$(k-3)1, 2$							1	1	0	0	2	2	6	
$(k-3)1, 1^2$							-1	1	0	0	-2	0	6	
$k-4, 4$									1	1	1	1	1	
$k-4, 31$									-1	0	-1	1	3	
$k-4, 2^2$									0	-1	2	0	2	
$k-4, 21^2$									1	0	-1	-1	3	
$k-4, 1^4$									-1	1	1	-1	1	

$n=4, r=3$	0	1	0	0	0	0	0	0	0	0	0	$\mu^{(3)}$
	0	0	2	1^2	1	1	0	0	0	0	0	$\mu^{(2)}$
$\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$	1	1	0	0	2	1^2	4	31	2^2	21^2	1^4	$\mu^{(1)}$
$k-1, 1, 0$	k	2	1	1	1	3	1	2	1	3	5	
$(k-2)1, 1, 0$		1	0	0	0	2	0	1	0	2	4	
$k-2, 0, 2$			1	1	1	1	1	0	3	1	3	
$k-2, 0, 1^2$			-1	1	1	1	-1	0	-1	1	3	
$k-3, 2, 1$					1	1	0	0	2	2	6	
$k-3, 1^2, 1$					-1	1	0	0	-2	0	6	
$k-4, 4, 0$							1	1	1	1	1	
$k-4, 31, 0$							-1	0	-1	1	3	
$k-4, 2^2, 0$							0	-1	2	0	2	
$k-4, 21^2, 0$							1	0	-1	-1	3	
$k-4, 1^4, 0$							-1	1	1	-1	1	

	1	0	0	0	0	0	0	0	0	0	0	$\mu^{(4)}$
$n=4, r>n$	0	1	0	0	0	0	0	0	0	0	0	$\mu^{(3)}$
	0	0	2	1^2	1	1	0	0	0	0	0	$\mu^{(2)}$
	0	1	0	0	2	1^2	4	31	2^2	21^2	1^4	$\mu^{(1)}$
$k-1, 0, 0, 0, 1$	1	1	1	1	1	1	1	1	1	1	1	
$k-2, 1, 0, 1, 0$		1	0	0	0	2	0	1	0	2	4	
$k-2, 0, 2$			1	1	1	1	1	0	3	1	3	
$k-2, 0, 1^2$			-1	1	1	1	-1	0	-1	1	3	
$k-3, 2, 1$					1	1	0	0	2	2	6	
$k-3, 1^2, 1$					-1	1	0	0	-2	0	6	
$k-4, 4, 0, 0$							1	1	1	1	1	
$k-4, 31, 0$							-1	0	-1	1	3	
$k-4, 2^2, 0$							0	-1	2	0	2	
$k-4, 21^2, 0$							1	0	-1	-1	3	
$k-4, 1^4, 0$							-1	1	1	-1	1	

If $r|r'$ then

$$GL_k \supset W_{r',k} \supset W_{r,k} \supset S_k$$

$$S_n \subset P_{n,r'} \subset P_{n,r} \subset P_n$$

For $w \in W_{r,k}, d_{\mu,n} \in P_{n,r'}$

$$k^{\frac{n-||\mu||}{r}} f_\mu(w) = \sum_\lambda \chi^\lambda_{W_{r,k}}(w) \chi^\lambda_{P_{n,r}(k)}(d_{\mu,n})$$

$$k^{\frac{n-||\mu||}{r'}} f_\mu(w) = \sum_\nu \chi^\nu_{W_{r',k}}(w) \chi^\nu_{P_{n,r'}(k)}(d_{\mu,n})$$

Therefore

if

$$\chi^\nu_{W_{r',k}} = \sum_\lambda c^\nu_\lambda \chi^\lambda_{W_{r,k}}$$

then

$$\chi^\lambda_{P_{n,r}(k)}(d_{\mu,n}) = k^{\frac{n-||\mu||}{r} - \frac{n-||\mu||}{r'}} \sum_\nu c^\nu_\lambda \chi^\nu_{P_{n,r'}(k)}(d_{\mu,n})$$