A combinatorial approach to doubly transitive permutation groups

Izumi Miyamoto
Department of Computer and Media Science,
University of Yamanashi,
Kofu 400-8511, Japan
imiyamoto@yamanashi.ac.jp

1 Introduction

Known doubly transitive permutation groups are well studied and the classification of all doubly transitive groups has been done by applying the classification of the finite simple groups. Readers may refer to [6]. This may mean that it is not still sufficient to study doubly transitive groups as permutation groups. Typical arguments on doubly transitive groups are seen in the book[3]. In the present paper we will give a combinatorial approach to doubly transitive groups. We use a combinatorial structure called a superscheme defined by the stabilizer of a point in a doubly transitive group and try to construct the superscheme defined by the given doubly transitive group. So if a superscheme defined by a transitive group is given first, our algorithm will construct the superscheme defined by the expected transitive extension of the given group. We do not use any group elements in this construction. Then if no superscheme is constructed, we can conclude that there is no transitive extension. If a superscheme is constructed, its automorphism group can be expected to be closely related to the transitive extension. However we will not compute the automorphism group, since we can guess what doubly transitive group it is. We will show how our algorithm works in the case of projective special linear groups $PSL(m,q)$, $m \geq 3$. Our algorithm also works successfully in the case of symplectic groups $Sp(2m,2)$, $m \geq 3$, over $GF(2)$ acting on the cosets by $O^+(2m,2)$ and $O^-(2m,2)$. In PSL-cases the automorphism groups of the obtained superschemes are usually imagined to be PGL. We also compute the transitive extensions of these groups itself. Our algorithm shows that they do not have transitive extensions except that $PSL(m,2)$ and $PSL(3,4)$, which are known to give $AGL(m,2)$ and the Mathieu group M_{22} of degree 22, and so on, we can consider the extensions of the obtained groups. Computer experiments suggest that our algorithm can be applied to any doubly or more transitive groups. We note that an obtained superscheme may not have a doubly transitive automorphism group.

Association schemes are often used to study permutation groups. A doubly transitive group of degree n defines a trivial association scheme which is the same one given by the symmetric group of degree n. So it is worthless to consider the association schemes given by doubly transitive groups. If a group G is doubly transitive and not triply transitive, the stabilizer of a point in G defines a non-trivial association scheme. But as is mentioned in [7], this association scheme is not sufficient to construct the group G. I will repeat about this fact briefly in the present paper. So we will use superschemes to construct doubly transitive groups from their stabilizers of a point. Superschemes are introduced in [5, 8]. We will follow a slightly different definition of superschemes given in [4]. In particular association schemes are superschemes.

We note that we used `GAP`[2] for our computer experiments.

2 Association schemes and superschemes

Let $X = \{x_1,\ x_2, \cdots, x_s\}$ be the set of vertices. Then an association scheme $(X, \{R_i\}_{0\leq i\leq d})$ and a superscheme (X, Π) are defined as follows.

Definition. $(X, \{R_i\}_{0\leq i\leq d})$ is an *association scheme* if and only if

1. $R_0 = \{(x, x)|x \in X\}$,
2. $\{R_0, R_1, \cdots, R_d\}$ is a partition of $X \times X$,
3. for all R_i there exists $R_{i'}$ such that $R_{i'} = \{(y, x)|(x, y) \in R_i\}$,
4. for all R_i, R_j, R_k and for all $(x, y) \in R_k$, there exists a constant number p_{ijk} such that
$$p_{ijk} = \#\{z \in X|(x, z) \in R_i, (z, y) \in R_j, (x, y) \in R_k\}.$$

Definition. (X, Π) is a *superscheme* if and only if

1. $\Pi = \{\Pi^1, \Pi^2, \cdots, \Pi^t\}$ for some $t \geq 2$, Π^l is a partition of X^l for $1 \leq l \leq t$,
2. let $\sigma((y_1, y_2,\ \cdots, y_l)) = (y_{\sigma(1)}, y_{\sigma(2)}, \cdots,\ y_{\sigma(l)})$ for $\sigma \in Sym(l)$, let $\Pi^l = \{R^l_0, R^l_1, \cdots, R^l_{d_l}\}$, $1 \leq l \leq t$, then $\sigma(R^l_k) \in \Pi^l$ for all R^l_k and all $\sigma \in Sym(l)$,
3. let a projection $\pi^l : X^l \to X^{l-1}$ be defined by $\pi^l((y_1, y_2, \cdots, y_{l-1}, y_l)) = (y_1, y_2, \cdots, y_{l-1})$, then $\pi^l(R^l_k) \in \Pi^{l-1}$ for all $R^l_k \in \Pi^l$, $2 \leq l \leq t$,
4. for all R^l_k, $2 \leq l \leq t$, for all $y = (y_1, y_2, \cdots, y_{l-1}) \in \pi^l(R^l_k)$, there exists a constant number p^l_k such that $p^l_k = |(\pi^l)^{-1}(y) \cap R^l_k|$. In particular $p^l_k = |R^l_k|/|\pi^l(R^l_k)|$.

For more properties of association schemes, readers may refer to [1]. The properties 2 and 4 of a superscheme are called *symmetric* and *regular* in [4], respectively. Referring to the number t in the definition of a superscheme, we simply call a t-superscheme. Each R^l_i is called a *relation*.

By the property 4 an association scheme always induces a 3-superscheme such that $\Pi^1 = \{X\}$, $\Pi^2 = \{R_0, R_1, \cdots, R_d\}$ and Π^3 consists of $R_{i,j,k} = \{(x, y, z)| \ (x, y) \in R_k, (x, z) \in R_i, (z, y) \in R_j\}$, where some of them may be empty.

3 Computation of superschemes related to transitive extensions

Let G be a transitive permutation group on a set X and let G_α be the stabilizer of a point $\alpha \in X$ in G. If the group G_α is given first, then the group G is said to be the transitive extension of G_α. Suppose that G is $(t-1)$-ply transitive and not t-ply transitive on X, where $t \geq 3$. Suppose $X = \{1, 2, \cdots, n, n+1\}$. Let $X^{(l)} = \{(i_1, i_2, \cdots, i_l)|i_j \in X, \text{all } i_j \text{ are distinct}\}$ for $1 \leq l \leq t$. Then the orbits of the stabilizer G_{n+1} of $n+1 \in X$ in G acting on X^l are obtained from those on $X^{(l)}$. So we will only consider the orbits on $X^{(l)}$ and consider the partition of $X^{(l)}$ for superschemes, which will be denoted so that $\Pi^{(l)} = \{R_1^{(l)}, \cdots, R_{d_l}^{(l)}\}$. We set $\pi_j^l = \pi^l \cdot (j, j+1, \cdots, l)$, where $(j, j+1, \cdots, l) \in Sym(l)$. Then by the symmetricity and the regularity of superschemes we have a constant number $p_{k,j}^l$ such that $p_{k,j}^l = |(\pi_j^l)^{-1}(y) \cap R_k^l|$ for all R_k^l, $2 \leq l \leq t$ and for all $y = (y_1, y_2, \cdots, y_{l-1}) \in \pi_j^l(R_k^l)$.

Let (X, Π') be the t-superscheme given by a $(t-2)$-ply transitive group G_{n+1} on $X \backslash \{n+1\}$. Suppose that there exists the transitive extension G of G_{n+1} and let (X, Π) be the t-superscheme given by G. Then the superscheme given by a $(t-1)$-ply transitive group satisfies that $\Pi^{(l)} = \{X^{(l)}\}$ for $1 \leq l < t$. Now we try to combine the orbits of G_{n+1} on $X^{(t)}$ to be the possible orbits of G on $X^{(t)}$. This is done only using the superschemes. We will construct all the possible $\Pi^{(t)}$ from $\Pi'^{(t)}$. Since G is $(t-1)$-ply transitive, the following holds.

Proposition 1 *Let $R_k'^{(t-1)}$, $1 \leq k \leq r$, be the relations in $\Pi'^{(t-1)}$ such that $R_k'^{(t-1)} \subseteq (X \backslash \{n+1\})^{(t-1)}$. Then G has r orbits on $X^{(t)}$ which are of size $|X| \cdot |R_k'^{(t-1)}|$, $k = 1, 2, \cdots, r$.*

Theorem 2 *Let $\Pi'^{(t)} = \{R_1', R_2', \cdots, R_d'\}$ and set $Y = \{1, 2, \cdots, d\}$. Let $\{Y_1, Y_2, \cdots, Y_r\}$ be a partition of Y and set R_k to be the union of R_i', $i \in Y_k$ for $1 \leq k \leq r$. If it hold that $\Pi^{(t)} = \{R_1, R_2, \cdots, R_r\}$, then $\pi_j^t(R_k) = X^{(t-1)}$ and we have the constants $p_{k,j}^t = |X| \cdot |R_k'^{(t-1)}|/|X^{(t-1)}| = |R_k'^{(t-1)}|/|(X \backslash \{n+1\})^{(t-2)}|$ for all $1 \leq k \leq r$ and all $1 \leq j \leq t$ in the t-superscheme (X, Π). Let $p_{i,j}'^t = |R_i'|/|\pi_j^t(R_i')|$ be the constants of (X, Π'). For any $R_s'^{(t-1)} \in \Pi'^{(t-1)}$ we set $Y_{k,j,s} = \{i \in Y_k | \pi_j^t(R_i') = R_s'^{(t-1)}\}$. Then we have $p_{k,j}^t = \sum_{i \in Y_{k,j,s}} p_{i,j}'^t$.*

Theorem 2 is a little complicated to check the conditions without a computer and it is rather an algorithm which one can understand easily if one see the examples in the next section.

Table 1
Orbits of the stabilizer of P_l in $PSL(m,q)$ on $\mathbf{P}^{(2)}$ and those of $PSL(m,q)$ on $(\mathbf{P}\cup\{P_0\})^{(3)}$

	stabilizer of P_l		$PSL(m,q)$		
No.	size	rep.	size	rep.	property
1	$n(q-1)$	(P_i,P_j)	$(n+1)n(q-1)$	(P_i,P_j,P_l)	$P_j\in\langle P_i,P_l\rangle$
2	$n(n-q)$	(P_i,P_j)	$(n+1)n(n-q)$	(P_i,P_j,P_l)	$P_j\notin\langle P_i,P_l\rangle$
3	n	(P_l,P_i)	$(n+1)n$	(P_0,P_i,P_l)	
4	n	(P_i,P_l)	$(n+1)n$	(P_i,P_0,P_l)	
5			$(n+1)n$	(P_i,P_l,P_0)	

4 Examples

We will show how our algorithm works for $G=PSL(m,q)$, $m\ge 3$. Let G_{n+1} be the stabilizer of a point P_l in G acting on the projective space $\mathbf{P}$ of dimension $m-1$ over a finite field of q elements. So $X=\mathbf{P}$, $t=3$ and $n=q+q^2+\cdots+q^{m-1}$. The orbits of G_{n+1} on $\mathbf{P}^{(2)}$ are shown in Table 1 together with those of G on $(\mathbf{P}\cup\{P_0\})^{(3)}$. G_{n+1} has four orbits on $\mathbf{P}^{(2)}$ and two of them are contained in $(\mathbf{P}\backslash\{P_l\})^{(2)}$, which are numbered 1 and 2 in Table 1. Table 1 shows that, for instance, the orbit 1 is of size $n(q-1)$ and consists of the couples (P_i,P_j) such that the point P_j belongs to the projective line $\langle P_i,P_l\rangle$. Then by Theorem 2 G has two orbits on $\mathbf{P}^{(3)}$ and we have $p^3_{j,1}=q-1$ and $p^3_{j,2}=n-q$, $1\le j\le 3$.

Table 2 consists of two parts. The first part of table 2 shows the orbits of G_{n+1} on $\mathbf{P}^{(3)}$. For instance, G_{n+1} has $q-2$ orbits $1(1),\cdots,1(q-2)$ which consist of (P_i,P_j,P_k) satisfying $P_j,P_k\in\langle P_l,P_i\rangle$. G_{n+1} is transitive on the triples (P_i,P_j,P_k) satisfying $P_k\in\langle P_l,P_i,P_j\rangle$ and any three of P_i,P_j,P_k and P_l non collinear if $m\neq 3$ or $3\not| q-1$, which is denoted by orbit 6. If $m=3|q-1$, then G_{n+1} has three orbits on them. So in this case they are denoted by 6(1), 6(2) and 6(3). The second part of Table 2 denotes the orbit numbers s of G_{n+1} on $\mathbf{P}^{(2)}$ in Table 1 such that $\pi^t_j(R'_i)=R'^{(t-1)}_s$ in the columns of label "im." and the constants $p'^3_{i,j}=|R'_i|/|\pi^3_j(R'_i)|$ in the columns of label "mult." for each orbit shown in the first part of Table 2. Here $\pi_j=\pi^3_j$, $1\le j\le 3$.

Now we compute the partition $\{Y_1,Y_2\}$ of the set Y of the orbit numbers in the first part of Table 2. We have $\{2,3,4,9,11,13\}\subset Y_2$, since $n-q>q-1$ and each of the orbits $\{2,3,4,9,11,13\}$ has a constant $p'^3_{i,j}=n-q$ for some i, j. Then $\{8,10,12\}\subset Y_1$. Furthermore we have $\{1(1),1(2),\cdots,1(q-2)\}\subset Y_1$, since $n-q=p^3_{2,1}=p'^3_{4,1}$ implies $Y_{2,1,1}=\{4\}$. Here we consider three cases. In the first case we assume that $q>2$ and ($m\neq 3$ or $q\neq 4$). We assume $q=2$ in the second case and assume $m=3$ and $q=4$ in the third case.

In the first case $p'^3_{6,1}=(q-1)^2>q-1$ or $p'^3_{6(i),1}=(q-1)^2/3>q-1$. So we have 6 or $6(i)\in Y_2$. If $m=3$, there does not exist the orbit 7, as $n=q+q^2$, else $p'^3_{7,1}>q-1$. So we have $7\in Y_2$ in general. Thus we have $Y_1=\{1(1,\cdots,q-2),5,8,10,12\}$ and $Y_2=Y\backslash Y_1$. Here we see, for instance, in the π_1-column of Table 2 that $Y_{1,1,1}=\{1(1,\cdots,q-2),8\}$, $Y_{1,1,2}=\{5\}$,

Table 2

Orbits of the stabilizer of P_l in $PSL(m, q)$ on $\mathbf{P}^{(3)}$

No.	size	rep.	property
1(1)	$n(q-1)$	(P_i, P_j, P_k)	$P_j, P_k \in \langle P_l, P_i \rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1(q-2)$	$n(q-1)$	(P_i, P_j, P_k)	$P_j, P_k \in \langle P_l, P_i \rangle$
2	$n(q-1)(n-q)$	(P_i, P_j, P_k)	$P_j \in \langle P_l, P_i \rangle, P_k \notin \langle P_l, P_i \rangle$
3	$n(q-1)(n-q)$	(P_i, P_j, P_k)	$P_j \notin \langle P_l, P_i \rangle, P_k \in \langle P_l, P_i \rangle$
4	$n(q-1)(n-q)$	(P_i, P_j, P_k)	$P_j \notin \langle P_l, P_i \rangle, P_k \in \langle P_l, P_j \rangle$
5	$n(q-1)(n-q)$	(P_i, P_j, P_k)	$P_j \notin \langle P_l, P_i \rangle, P_k \in \langle P_i, P_j \rangle$
6	$n(q-1)^2(n-q)$	(P_i, P_j, P_k)	$P_k \in \langle P_l, P_i, P_j \rangle$, non collinear
$6(1,2,3)$	$n(q-1)^2(n-q)/3$	(P_i, P_j, P_k)	as above and $m = 3 \mid q-1$
7	$n(n-q)(n-q-q^2)$	(P_i, P_j, P_k)	$P_k \notin \langle P_l, P_i, P_j \rangle$, non collinear
8	$n(q-1)$	(P_l, P_i, P_j)	$P_j \in \langle P_l, P_i \rangle$
9	$n(n-q)$	(P_l, P_i, P_j)	$P_j \notin \langle P_l, P_i \rangle$
10	$n(q-1)$	(P_i, P_l, P_j)	$P_j \in \langle P_l, P_i \rangle$
11	$n(n-q)$	(P_i, P_l, P_j)	$P_j \notin \langle P_l, P_i \rangle$
12	$n(q-1)$	(P_i, P_j, P_l)	$P_j \in \langle P_l, P_i \rangle$
13	$n(n-q)$	(P_i, P_j, P_l)	$P_j \notin \langle P_l, P_i \rangle$

The properties of projections

	π_1		π_2		π_3	
No.	im.	mult.	im.	mult.	im.	mult.
$1(1, \cdots, q-2)$	1	1	1	1	1	1
2	2	$q-1$	2	$q-1$	1	$n-q$
3	2	$q-1$	1	$n-q$	2	$q-1$
4	1	$n-q$	2	$q-1$	2	$q-1$
5	2	$q-1$	2	$q-1$	2	$q-1$
6	2	$(q-1)^2$	2	$(q-1)^2$	2	$(q-1)^2$
6(1,2,3)	2	$(q-1)^2/3$	2	$(q-1)^2/3$	2	$(q-1)^2/3$
7	2	$n-q-q^2$	2	$n-q-q^2$	2	$n-q-q^2$
8	1	1	3	$q-1$	3	$q-1$
9	2	1	3	$n-q$	3	$n-q$
10	3	$q-1$	1	1	4	$q-1$
11	3	$n-q$	2	1	4	$n-q$
12	4	$q-1$	4	$q-1$	1	1
13	4	$n-q$	4	$n-q$	2	1

$Y_{1,1,3} = \{10\}$, $Y_{1,1,4} = \{12\}$, $Y_{2,1,1} = \{4\}$, $Y_{2,1,2} = \{2,3,6,7,9\}$, $Y_{2,1,3} = \{11\}$ and $Y_{2,1,4} = \{13\}$.

In the second case, since $(q-1)^2 = q-1$, the orbits 5 and 6 have the same entries. Since $q-2 = 0$, there exist no orbits $1(1,\cdots,q-2)$. So we have two possibilities $Y_1 = \{5,8,10,12\}$ and $Y_1 = \{6,8,10,12\}$. Then $Y_2 = Y \backslash Y_1$, respectively. Computer experiment shows that the automorphism groups of the obtained superschemes are $PSL(3,2)$ in both of the possibilities if $m = 3$.

In the third case, since $(q-1)^2/3 = q-1$, all the orbits 5 and 6(1,2,3) have the same entries. So we have four possibilities $Y_1 = \{1(1,\cdots,q-2), i, 8, 10, 12\}$, where i is any of $\{5, 6(1), 6(2), 6(3)\}$, and $Y_2 = Y \backslash Y_1$. In all of the possibilities computer experiment gives that the automorphism groups are $P\Gamma L(3,4)$.

Here we mention about the association scheme defined by G_{n+1} on $X\backslash\{n+1\}$. In an association scheme we easily see that the orbits of G_{n+1} on $(X\backslash\{n+1\})^{(3)}$ which have the same entries in the "im."-columns are contained in a common relation. So in particular the orbits 5 and 6 are contained in a relation of the association scheme defined by G_{n+1}. But the above argument gives that the orbits 5 and 6 belong to the different sets of Y_1 and Y_2 with $Y_1 \cap Y_2 = \phi$. Therefore association schemes are not sufficient to compute transitive extensions.

Next we try to compute the 4-superschemes given by the transitive extensions of $PSL(m,q)$ itself. So $t = 4$ and $Y = \{1,2,\cdots,15\}$. Consulting Table 3, similar arguments as above give that $\{2,3,4,5\} \subset Y_2$. Here we notice that the four orbits 2, 3, 4, and 5 are permuted by the $Sym(4)$ in the property 2 of a superscheme with $t = 4$. We consider three cases as above. In the first case the same argument as above implies $5 \in Y_1$. But already $5 \in Y_2$, a contradiction. So we have no possibilities in the first case. In the second case we have one possibility $Y_1 = \{6,8,10,12,14\}$. So in this case we may have the transitive extension $AGL(m,2)$. In the third case we have three possibilities $Y_1 = \{1(1,\cdots,q-2), i, 8, 10, 12, 14\}$, where i is any of $\{6(1), 6(2), 6(3)\}$. So we can expect the transitive extension M_{22}.

Then let us go forward to consider the further transitive extensions. Now $t = 5$ and $Y = \{1,2,\cdots,17\}$. Comparing Table 2 and 3, readers may easily guess the properties of the orbits 16 and 17. Then as we noticed above, in this extension the $Sym(5)$ should permute the orbits 2, 3, 4, 5 and one more orbit which should be the orbit 6 or 6(i). This gives some conditions that π_5 should satisfy. This also implies that they are all contained in Y_2, since the $Sym(5)$ acts trivially on $\Pi^{(5)} = \{R_1^{(5)}, R_2^{(5)}\}$ with $|R_1^{(5)}| \neq |R_2^{(5)}|$. So if $q = 2$, we have $Y_1 = \{8,10,12,14,16\}$, which does not satisfy the conditions in Theorem 2. Thus $AGL(m,2)$ does not have a transitive extension. If $m = 3$ and $q = 4$, then we may assume $6(3) \in Y_2$ and we have still two possibilities $Y_1 = \{1(1,\cdots,q-2), i, 8, 10, 12, 14, 16\}$, where i is any of $\{6(1), 6(2)\}$. Here we expect M_{23}. In the next extension we can expect M_{24} with $t = 6$, $|Y| = 19$, and $Y_1 = \{1(1,\cdots,q-2), 6(1), 8, 10, 12, 14, 16\}$. At last we consider the transitive extension of M_{24}. Then $t = 7$, $|Y| = 21$ and the $Sym(7)$ should permute the orbits 2, 3, 4, 5, 6(1), 6(2) and 6(3). Hence they are contained in Y_2. So we

Table 3

Orbits of $PSL(m,q)$ on $(\mathbf{P} \cup \{P_0\})^{(4)}$

No.	size ($a = (n+1)n$)	rep.	property
1(1)	$a(q-1)$	(P_i, P_j, P_k, P_l)	$P_j, P_k \in \langle P_l, P_i \rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1(q-2)$	$a(q-1)$	(P_i, P_j, P_k, P_l)	$P_j, P_k \in \langle P_l, P_i \rangle$
2	$a(q-1)(n-q)$	(P_i, P_j, P_k, P_l)	$P_j \in \langle P_l, P_i \rangle, P_k \notin \langle P_l, P_i \rangle$
3	$a(q-1)(n-q)$	(P_i, P_j, P_k, P_l)	$P_j \notin \langle P_l, P_i \rangle, P_k \in \langle P_l, P_i \rangle$
4	$a(q-1)(n-q)$	(P_i, P_j, P_k, P_l)	$P_j \notin \langle P_l, P_i \rangle, P_k \in \langle P_l, P_j \rangle$
5	$a(q-1)(n-q)$	(P_i, P_j, P_k, P_l)	$P_j \notin \langle P_l, P_i \rangle, P_k \in \langle P_i, P_j \rangle$
6	$a(q-1)^2(n-q)$	(P_i, P_j, P_k, P_l)	$P_k \in \langle P_l, P_i, P_j \rangle$, non collinear
$6(1,2,3)$	$a(q-1)^2(n-q)/3$	(P_i, P_j, P_k, P_l)	as above and $m = 3 \mid q-1$
7	$a(n-q)(n-q-q^2)$	(P_i, P_j, P_k, P_l)	$P_k \notin \langle P_l, P_i, P_j \rangle$, non collinear
8	$a(q-1)$	(P_0, P_i, P_j, P_k)	$P_k \in \langle P_i, P_j \rangle$
9	$a(n-q)$	(P_0, P_i, P_j, P_k)	$P_k \notin \langle P_i, P_j \rangle$
10	$a(q-1)$	(P_i, P_0, P_j, P_k)	$P_k \in \langle P_i, P_j \rangle$
11	$a(n-q)$	(P_i, P_0, P_j, P_k)	$P_k \notin \langle P_i, P_j \rangle$
12	$a(q-1)$	(P_i, P_j, P_0, P_k)	$P_k \in \langle P_i, P_j \rangle$
13	$a(n-q)$	(P_i, P_j, P_0, P_k)	$P_k \notin \langle P_i, P_j \rangle$
14	$a(q-1)$	(P_i, P_j, P_k, P_0)	$P_k \in \langle P_i, P_j \rangle$
15	$a(n-q)$	(P_i, P_j, P_k, P_0)	$P_k \notin \langle P_i, P_j \rangle$

The properties of projections

	π_1		π_2		π_3		π_4	
No.	im.	mult.	im.	mult.	im.	mult.	im.	mult.
$1(1, \cdots, q-2)$	1	1	1	1	1	1	1	1
2	2	$q-1$	2	$q-1$	1	$n-q$	2	$q-1$
3	2	$q-1$	1	$n-q$	2	$q-1$	2	$q-1$
4	1	$n-q$	2	$q-1$	2	$q-1$	2	$q-1$
5	2	$q-1$	2	$q-1$	2	$q-1$	1	$n-q$
6	2	$(q-1)^2$	2	$(q-1)^2$	2	$(q-1)^2$	2	$(q-1)^2$
6(1,2,3)	2	$(q-1)^2/3$	2	$(q-1)^2/3$	2	$(q-1)^2/3$	2	$(q-1)^2/3$
7	2	$n-q-q^2$	2	$n-q-q^2$	2	$n-q-q^2$	2	$n-q-q^2$
8	1	1	3	$q-1$	3	$q-1$	3	$q-1$
9	2	1	3	$n-q$	3	$n-q$	3	$n-q$
10	3	$q-1$	1	1	4	$q-1$	4	$q-1$
11	3	$n-q$	2	1	4	$n-q$	4	$n-q$
12	4	$q-1$	4	$q-1$	1	1	5	$q-1$
13	4	$n-q$	4	$n-q$	2	1	5	$n-q$
14	5	$q-1$	5	$q-1$	5	$q-1$	1	1
15	5	$n-q$	5	$n-q$	5	$n-q$	2	1

have $Y_1 = \{1(1, \cdots, q-2), 8, 10, 12, 14, 16, 18, 20\}$, which does not satisfy the conditions in Theorem 2. Thus M_{24} does not have a transitive extension.

For the cases of $Sp(2m, 2)$ acting on $Sp(2m, 2)/O^+(2m, 2)$ and $Sp(2m, 2)/O^-(2m, 2)$ we have the 3-superschemes defined by $Sp(2m, 2)$ and have no 4-superschemes which may give the transitive extensions of $Sp(2m, 2)$ itself. So in particular we have $Sp(2m, 2)$, $m \geq 3$ have no transitive extensions.

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